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Exact Solution for the Most General Minimally Coupled One-Dimensional Lattice Gauge Theories

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We consider one-dimensional lattice gauge theories constructed by the minimal coupling prescription. It is shown that these theories are exactly solvable in the thermodynamic limit. After considering the most general case, we discuss some special cases on finite lattices and also work out some examples. There is no phase transition in these minimally coupled theories.

INTRODUCTION

During the last two decades lattice gauge theories have been extensively studied (Wilson, 1974; Wegner, 1972; Kogut, 1979; Balian *et al.*, 1974). Lattice theories have no ultraviolet divergences, they provide a nonperturbative approach to some theories, such as QCD (e.g., Wilson, 1974), and they are theoretically interesting in themselves. They introduce possibilities which are absent in a continuum; for example, one can consider discrete gauge groups as well as continuous ones. So far, the main interest has been the study of lattice gauge theories (especially pure gauge theories) on multidimensional lattices (Wegner, 1972; Balian *et al.*, 1975). One cannot, however, consider the most general gauge theories on such lattices.

In the case of one-dimensional lattices, there is a dramatic change: there is only one Wilson loop (in closed lattices). So one can consider the general form of gauge-invariant interactions, including matter fields as well as gauge fields. We will see that these theories are all exactly solvable in the thermodynamic limit. An example is the Kazakov–Migdal induced gauge theory in one dimension studied by Caselle *et al.* (1992). To be more specific, we consider gauge-invariant Hamiltonians with compact gauge

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groups, and show that a statistical system having such an interaction is exactly solvable in the thermodynamic limit (Section 1). We further show that in this limit the gauge degrees of freedom decouple from the matter degrees of freedom (Section 1). Then we consider more special cases where the matter part of the interaction becomes a noninteracting theory and observables become uncorrelated (Section 2). In some cases one can also compute things for finite lattices (Section 3). In Section 4 some examples are presented, and in Section 5 we consider double gauge field theories. These are all examples of exactly solvable nonlocal interactions. There is no phase transition in these theories, even at zero temperature. However, in a future paper, we will study a generalized version of these theories which does have nontrivial phase structure.

0.1. Gauge Theory on Lattices

Consider a lattice consisting of a given set of sites i and links $\langle ij \rangle$. Next consider two sets V and $\tilde{V}$, a function $\tilde{}: V \to \tilde{V}$, and a multiplication from $\tilde{V} \times V$ to $\tilde{V}V$. Defining the matter field S on sites, one can write the Hamiltonian for a nearest-neighbor interaction as

$$H_0 = -\sum_{\langle ij \rangle} F(\tilde{S}_i S_j) \tag{0.1}$$

where F is a real-valued function.

Now, suppose that a group G acts on the sets V and $\tilde{V}$ through

$$S \to \hat{g} S \tag{0.2}$$

$$\widetilde{(\hat{g}S)} = \tilde{S}\hat{g}^{-1} \tag{0.3}$$

where $\hat{g}$ is a representation of g. Introducing a group element-valued field defined on links, one reaches a gauge-invariant Hamiltonian

$$H_{\mathrm{m}} = -\sum_{\langle ij \rangle} F(\tilde{S}_i \hat{U}_{\langle ij \rangle} S_j) \tag{0.4}$$

This Hamiltonian is invariant under local gauge transformation (Balian *et al.*, 1974)

$$\begin{aligned} S_i &\to \hat{g}_i S_i \\ U_{\langle ij \rangle} &\to g_i U_{\langle ij \rangle} g_j^{-1} \end{aligned} \tag{0.5}$$

To this (matter field) Hamiltonian one can add another function, which is a conjugation-invariant function (class function) of Wilson loops (Balian *et *al.*, 1974)

$$H = -\sum_{\langle ij\rangle} F(\tilde{S}_i \hat{U}_{\langle ij\rangle} S_j) - E(W_{l_1}, W_{l_2}, \ldots)$$
$$=: H_m + H_G \tag{0.6}$$

where the W_l are Wilson loops.

Observables of this theory are of two kinds: gauge-invariant paths $(\tilde{S}_i W_{i\ldots j} S_j)$, where $W_{i\ldots j}$ is a Wilson path starting from i and ending in j, and class functions of Wilson loops. Our task is to consider the partition function Z and correlation function $\langle\Omega\rangle$ of a statistical system having an interaction of the form (0.6). These are defined as

$$Z := \int \Big(\prod_i dS_i\Big)\Big(\prod_{\langle ij\rangle} dU_{\langle ij\rangle}\Big) \exp[-\beta H(\{S_i\}, \{U_{\langle ij\rangle}\})] \tag{0.7}$$

$$\langle \Omega(\{S_i\}, \{U_{\langle ij\rangle}\})\rangle := \frac{1}{Z}\int \Big(\prod_i dS_i\Big)\Big(\prod_{\langle ij\rangle} dU_{\langle ij\rangle}\Big)$$
$$\times \exp[-\beta H(\{S_i\}, \{U_{\langle ij\rangle}\})]\Omega(\{S_i\}, \{U_{\langle ij\rangle}\}) \tag{0.8}$$

In both cases the integration symbol is formal and may be integration or summation, according to whether we have a discrete or continuous set. The measure of the group is the invariant measure, and the measure of the matter field is also invariant under the action of the group.

0.2. One-Dimensional Lattice Gauge Theory

A one-dimensional open lattice has no Wilson loop, and a closed one has only one independent Wilson loop. So the Hamiltonian (0.6) is highly restricted and we have (for closed lattices)

$$H = -\sum_{i=1}^{N} F(\tilde{S}_i \hat{U}_{i+1/2} S_{i+1}) - E\Big(\prod_{i=1}^{N} U_{i+1/2}\Big) \tag{0.9}$$

where

$$X_{N+k} := X_k \tag{0.10}$$

and

$$U_{i+1/2} := U_{\langle i,i+1\rangle} \tag{0.11}$$

In one dimension, the number of gauge degrees of freedom is exactly equal to the number of gauge transformations. So it seems possible to eliminate the gauge field by suitable gauge fixing. This is almost the case: for open lattices this can be done and the Hamiltonian then reduces to H_0. So, for open lattices gauging has no effect.

If the lattice is closed, however, one cannot trivialize the Wilson loop by gauge transformation. In fact, by a suitable gauge transformation,

$$g_i = \prod_{k=1}^{i} U_{k-1/2}, \qquad i \leq i \leq N \tag{0.12}$$

$$U_{i+1/2} \to \begin{cases} 1, & i \neq 0 \\ \prod_{k=1}^{N} U_k - 1/2, & i = 0 \end{cases} \tag{0.13}$$

So the gauge field can be absorbed neither from the matter Hamiltonian nor from the gauge part. However, we will show that, if the gauge group G is compact, in the thermodynamic limit ($N \to \infty$) the Wilson loop has no effect on the matter Hamiltonian and the partition function correlators can be factorized.

1. GENERAL RESULTS

First, let us prove the factorizability of the partition function and correlators. To do so, first consider a case where G acts transitively on V (that is, V consists of a single orbit of G). Throughout this argument we assume that the group G is compact. We want to prove that, in the thermodynamic limit, the partial partition function

$$Z_m := \int \left(\prod_i dS_i \right) \exp\left[\sum_i f(\tilde{S}_i \hat{U}_{i+1/2} S_{i+1}) \right] \tag{1.1}$$

is independent of the $U_{i+1/2}$ ($f := \beta F$). Defining the transfer operator $P(U)$ as

$$[\psi P(U)](S) := \int dS' \, \psi(S') \exp[F(\tilde{S}' \hat{U} S)] \tag{1.2}$$

it is obvious that

$$Z_m = \operatorname{tr}\left[\prod_i P(U_{i+1/2}) \right] \tag{1.3}$$

Now consider the following lemma

Lemma. The eigenvector corresponding to the largest eigenvalue of the transfer operator $P(U)$ is independent of U.

Proof. From (1.2) we have

$$[\omega P(U)](S) \leq \max\{\psi(S')\} \int dS' \exp[f(\tilde{S}' \hat{U} S)] \tag{1.4}$$

and, as the integration measure is invariant under the action of G, the integral in the right-hand side of (1.4) does not depend on U or S. So if ψ obtains its maximum at $S_{\max}$, we have

$$[\psi P(U)](S) \le \mu\psi(S_{\max}) \tag{1.5}$$

where μ is a constant:

$$\mu := \int dS' \exp[(\tilde{S}'\hat{U}S)] \tag{1.6}$$

Now suppose that ψ is an eigenvector of $P(U)$ with eigenvalue λ. That is,

$$[\psi P(U)](S) = \lambda\psi(S) \tag{1.7}$$

We then have

$$\lambda\psi(S_{\max}) \le \mu\psi(S_{\max}) \tag{1.8}$$

One can always choose ψ so that $\psi(S_{\max})$ is positive. Then, from (1.8),

$$\lambda \le \mu \tag{1.9}$$

and equality holds iff we have equality in (1.4), that is, iff

$$\psi(S) = \text{const} \tag{1.10}$$

Therefore the eigenvector of $P(U)$ corresponding to its largest eigenvalue is the constant function, which does not depend on U. The largest eigenvalue does not depend on U either.

Note that in the proof we have used the compactness of G, and hence V, to guarantee the existence of a maximum for a continuous real-valued function on V. We also notice that the largest eigenvalue μ, defined through (1.6), is indeed finite.

Using the above lemma, one can deduce that, in the thermodynamic limit,

$$Z_{\mathrm{m}} = \mu^N \tag{1.11}$$

So (setting $e := \beta E$)

$$\begin{aligned} Z &= \mu^N \int \left(\prod_i dU_{i+1/2}\right) \exp\left[e\left(\prod_i U_{i+1/2}\right)\right] \\ &= [\mu \operatorname{vol}(G)]^N \frac{1}{\operatorname{vol}(G)} \int dU \exp[e(U)] \\ Z &= [\mu \operatorname{vol}(G)]^N v \end{aligned} \tag{1.12}$$

where

$$\mathrm{vol}(G) = \int dU \tag{1.13}$$

$$\nu := \frac{1}{\mathrm{vol}(G)} \int dU \exp[e(U)] \tag{1.14}$$

From now on we will use a normalization for the group measure such that the volume of the group becomes unity.

This argument can be readily generalized to cases where V is not a single orbit of G. In such a case we have

$$\begin{aligned} Z_{\mathrm{m}} &= \int \left(\prod_i dS_i\right) \exp\left[\sum_i f(\hat{S}_i \hat{U}_{i+1/2} S_{i+1})\right] \\ &= \int \left(\prod_i dS_i\right) \exp\left[\sum_i f(\hat{S}_i \hat{g}_i^{-1} \hat{U}_{i+1/2} \hat{g}_{i+1} S_{i+1})\right] \\ &= \int \left(\prod_i dS_i\right)\left(\prod_i dg_i\right) \exp\left[\sum_i f(\hat{S}_i \hat{g}_i^{-1} \hat{U}_{i+1/2} \hat{g}_{i+1} S_{i+1})\right] \end{aligned} \tag{1.15}$$

where we have used the invariance of dS_i under the action of the group. Now define a partial transfer operator $P_G(U, S, S')$ by

$$(\psi P_G)(g) := \int dg' \psi(g') \exp[f(\tilde{S}' \hat{g}^{-1} \hat{U} \hat{g} S)] \tag{1.16}$$

We then have

$$Z_{\mathrm{m}} = \int \left(\prod_i dS_i\right) \mathrm{tr}\left[\prod_i P_G(U_{i+1/2}, S_{i+1}, S_i)\right] \tag{1.17}$$

Similarly, the eigenvector corresponding to the largest eigenvalue of P_G is independent of U and depends only on the orbits of S and S', which we denote by $|S|$ and $|S'|$. So, in the thermodynamic limit we have

$$Z_{\mathrm{m}} = \int \left(\prod_{\mathrm{i}} dS_i\right)\left[\prod_i \mu(|S_i|, |S_{i+1}|)\right] \tag{1.18}$$

where μ is the largest eigenvalue of P_G, and

$$Z = \nu \int \left(\prod_i dS_i\right)\left[\prod_i \mu(|S_i|, |S_{i+1}|)\right] \tag{1.19}$$

What about the correlation functions? This argument still works if we are considering correlators of observables confined to a finite region. By a suitable gauge fixing, one can eliminate the gauge fields in that region.

Then the above arguments are valid and, in the thermodynamic limit, the integral

$$\int\left(\prod_i dS_i\right)\Omega_{\text{fin}}(\{S_i\},\{U_{i+1/2}\})\exp(-\beta H_{\text{m}})$$

is independent of the $U_{i+1/2}$. So we have

$$\begin{aligned}\langle\Omega_{\text{fin}}\rangle &= \frac{1}{vZ_m}\,v\int\left(\prod_i dS_i\right)\Omega_{\text{fin}}\exp(-\beta H_{\text{m}})\\ &= \frac{1}{Z_m}\int\left(\prod_i dS_i\right)\Omega_{\text{fin}}\exp(-\beta H_{\text{m}})\end{aligned}\tag{1.20}$$

and, as the right-hand sides are independent of the $U_{i+1/2}$,

$$\begin{aligned}\langle\Omega_{\text{fin}}\rangle &= \langle\Omega_{\text{fin}}\rangle_{H_{\text{m}}}\\ &= \langle\Omega^0_{\text{fin}}\rangle_{H_0}\end{aligned}\tag{1.21}$$

where by the subscripts H_{m} and H_0 we mean averaging with Boltzmann weights corresponding to H_{m} and H_0, respectively, and

$$\Omega^0_{\text{fin}}(\{S_i\},\{U_{i+1/2}\}) := \Omega_{\text{fin}}(\{S_i\},\{U_{i+1/2}=1\})\tag{1.22}$$

Now suppose that Ω is the product of Ω_G and Ω_{fin}, where Ω_G is a function of the Wilson loop. It is easily seen that

$$\begin{aligned}\langle\Omega\rangle &= \langle\Omega_G\rangle\langle\Omega_{\text{fin}}\rangle\\ &= \langle\Omega_G\rangle_{H_G}\langle\Omega^0_{\text{fin}}\rangle_{H_0}\end{aligned}\tag{1.23}$$

So we have proved the following result:

A one-dimensional gauge theory with nearest-neighbor interaction between matter fields on a closed lattice with a compact gauge group is (as long as we are considering observables which are either local or functions of the Wilson loop), in the thermodynamic limit, effectively decomposed into two noninteracting parts: A matter part, the Hamiltonian of which is H_m (or equivalently H_0), and a gauge part, which is a one-particle system.

2. MATTER FIELD SPACES CONSISTING OF A SINGLE ORBIT OF *G*

In this case, one can completely eliminate the matter field by suitable gauge fixing (even for finite lattices). In fact, the partial partition function

$$Z_p := \int\left(\prod_i dU_{i+1/2}\right)\exp\left[\prod_i f(\tilde{S}_i\hat{U}_{i+1/2}S_{i+1}) + e\left(\prod_i U_{i+1/2}\right)\right]\tag{2.1}$$

is independent of the S_i. To see this, suppose that we change S_k to

$$S'_k = \hat{g} S_k \tag{2.2}$$

leaving the other S_i unchanged. We then have

$$Z_p(S'_k) = \int \left(\prod_i dU'_{i+1/2} \right) \exp\left[\prod_i f(\tilde{S}_i \hat{U}'_{i+1/2} S_{i+1}) + e\left(\prod_i U'_{i+1/2} \right) \right] \tag{2.3}$$

where we have made the change of variable

$$U'_{k-1/2} = U_{k-1/2} g, \qquad U'_{k+1/2} = g^{-1} U_{k+1/2} \tag{2.4}$$

So

$$Z_p(\{S'_i\}) = Z_p(\{S_i\}) \tag{2.5}$$

It is easily seen that this argument can also be generalized to correlators. So we can eliminate the matter field and use the gauge-fixed Hamiltonian

$$H_{\mathrm{gf}} := -\sum_i F(\tilde{C} \hat{U}_{i+1/2} C) - E\left(\prod_i U_{i+1/2} \right) \tag{2.6}$$

where C is a constant member of V. Using the result of Section 1, we decompose the theory into two noninteracting parts (in the thermodynamic limit):

$$(H_{\mathrm{m}})_{\mathrm{gf}} := -\sum_i F(\tilde{C} \hat{U}_{i+1/2} C) \tag{2.7}$$

and H_G. But now there is no interaction in the matter part. So we conclude that, as long as we are considering local observables, the theory is free and essentially a one-particle theory. That is, observables at different points are uncorrelated, and of course distance-independent:

$$\langle \Omega_1 \Omega_2 \rangle = \langle \Omega_1 \rangle \langle \Omega_2 \rangle \tag{2.8}$$

if Ω_1 and Ω_2 depend on no common U_i. We also have

$$\langle \Omega \rangle = \langle \Omega \rangle_{(H_{\mathrm{m}})_{\mathrm{gf},r(\Omega)}} \tag{2.9}$$

where $(H_{\mathrm{m}})_{\mathrm{gf},r(\Omega)}$ is the sum of those terms of $(H_{\mathrm{m}})_{\mathrm{gf}}$ in which the U_i contributing in Ω enter. In the above arguments, Ω is a function of gauge-invariant paths with the substitution $S_i \to C$.

The total partition function takes the form

$$Z = \left\{ \left(\int dS \right) \int dU \exp[f(\tilde{C} \hat{U} C)] \right\}^N \int dU \exp[e(U)] \tag{2.10}$$

comparing this with (1.12), we find

$$\mathrm{vol}(V)\int dU\,\exp[f(\tilde{C}\hat{U}C)]=\mathrm{vol}(G)\int dS'\,\exp[f(\tilde{S}'\hat{U}S)] \tag{2.11}$$

So any correlation function can be obtained by a (finite) number of integrations. As an example, consider pure gauge correlators, that is, class functions of the Wilson loop. These functions are linear combinations of group characters (see the appendix); so one only needs to consider correlators of the form

$$\left\langle X_\mu\left(\prod_i U_{i+1/2}\right)\right\rangle=\frac{\int dU\,X_\mu(U)\,\exp[e(U)]}{\int dU\,\exp[e(U)]} \tag{2.12}$$

where X_μ is the character of the group in the (unitary) representation μ. Here $\exp[e(U)]$ itself is also a class function, so one can expand it as

$$\begin{aligned}\exp[e(U)]&=\exp[\beta E(U)]\\&=\sum_\lambda I_\lambda^{(G,E)}(\beta)X_\lambda(U)\end{aligned} \tag{2.13}$$

where the summation runs over irreducible unitary representations of G, and

$$I_\lambda^{(G,E)}(\beta)=\int dU\,X_\lambda(U^{-1})\,\exp[\beta E(U)] \tag{2.14}$$

(see the appendix). From these we obtain

$$\left\langle X_\mu\left(\prod_i U_{i+1/2}\right)\right\rangle=\frac{I_{\bar{\mu}}^{(G,E)}(\beta)}{I_0^{(G,E)}(\beta)} \tag{2.15}$$

where $\bar{\mu}$ is the complex conjugate representation of μ, and 0 is the trivial representation. There is a special case where one can go further and calculate correlators for finite lattices. We will consider this case in the following section.

3. CONJUGATION-INVARIANT MATTER HAMILTONIANS AND OBSERVABLES

As a special case of Section 2, consider a gauge-fixed matter Hamiltonian having the property

$$F(\tilde{C}\hat{U}C)=F(\tilde{C}\hat{g}^{-1}\hat{U}\hat{g}C) \tag{3.1}$$

If we restrict ourselves to observables which have the same property, that is, invariance under local conjugation, we can compute correlators for finite lattices. First we compute the partition function

$$Z = \int \left(\prod_i dU_{i+1/2}\right) \exp\left[\sum_i f_C(U_{i+1/2}) + e\left(\prod_i U_{i+1/2}\right)\right]$$

where

$$f_C(U) := f(\tilde{C}UC) \tag{3.2}$$

Using

$$\exp[\beta F_C(U)] = \sum_\lambda I_\lambda^{(G,R_C)}(\beta) X_\lambda(U) \tag{3.3}$$

we have

$$Z = \sum_{\{\lambda_i + 1/2\},\mu} \left\{ \int \left(\prod_i dU_{i+1/2}\right)\left[\prod_i X_{\lambda_{i+1/2}}(U_{i+1/2})\right] X_{\bar{\mu}} \right.$$

$$\left. \times \left(\prod_i U_{i+1/2}\right)\right\} I_{\lambda_{i+1/2}}^{(G,F_C)}(\beta) I_{\bar{\mu}}^{(G,E)}(\beta) \tag{3.4}$$

and (using the appendix)

$$\int \left(\prod_i dU_{i+1/2}\right)\left[\prod_i X_{\lambda_{i+1/2}}(U_{i+1/2})\right] X_{\bar{\mu}}\left(\prod_i U_{i+1/2}\right)$$

$$= \left(\prod_i \delta_{\lambda_{i+1/2},\mu}\right)[d(\mu)]^{1-N} \tag{3.5}$$

where $d(\mu)$ is the dimension of the representation μ. So

$$Z = \sum_\mu \left[\frac{I_\mu^{(G,F_C)}(\beta)}{d(\mu)}\right]^N I_{\bar{\mu}}^{(G,E)}(\beta)\, d(\mu) \tag{3.6}$$

Now, consider correlators of the form

$$\left\langle \prod_i X_{\sigma_{i+1/2}}(U_{i+1/2}) \right\rangle =: \langle\{\sigma_{i+1/2}\}\rangle \tag{3.7}$$

Every (conjugation-invariant) correlator can be expanded in terms of these. We have

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{1}{Z} \sum_{\{\lambda_i + 1/2\},\mu} \left\{ \int \left(\prod_i dU_{i+1/2}\right) \right.$$

$$\left. \times \left[\prod_i X_{\sigma_{i+1/2}}(U_{i+1/2}) X_{\lambda_{i+1/2}}(U_{i+1/2})\right] X_{\bar{\mu}}\left(\prod_i U_{i+1/2}\right)\right\}$$

$$\times I_{i+1/2}^{(G,F_C)}(\beta) I_{\bar{\mu}}^{(G,E)}(\beta) \tag{3.8}$$

Defining nonnegative integers $\{\rho, \tau; \phi\}$ through

$$[\rho] \otimes [\tau] = \bigoplus_{\phi} \{\rho, \tau; \phi\}[\phi] \tag{3.9}$$

where $[\rho]$, $[\tau]$, and $[\phi]$ are irreducible unitary representations of G, we conclude that

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{1}{Z} \sum_{\mu} \left\{ \prod_{i} \left[\frac{\sum_{\lambda_{i+1/2}} \{\sigma_{i+1/2}, \lambda_{i+1/2}; \mu\} I^{(G,F_C)}_{\lambda_{i+1/2}}(\beta)}{d(\mu)} \right] \right\} I^{(G,E)}_{\bar{\mu}}(\beta)\, d(\mu) \tag{3.10}$$

We see, as one expects from the very beginning, that these correlators are distance-independent. That is, they depend only on the set of $\sigma_{i+1/2}$, not on their ordering. One can also calculate the correlator for the characters of the Wilson loop. The calculation is similar to the above, and one obtains

$$\left\langle X_\sigma\left(\prod_i U_{i+1/2}\right)\right\rangle = \frac{1}{Z} \sum_{\mu,\lambda} \{\sigma, \mu; \bar{\lambda}\} I^{(G,E)}_{\mu}(\beta)\, d(\lambda) \left[\frac{I^{(G,F_C)}_{\lambda}(\beta)}{d(\lambda)}\right]^N \tag{3.11}$$

One can also easily go to the thermodynamic limit. Using

$$|X_\lambda(U)| \le d(\lambda) \qquad \text{if} \quad \lambda \neq 0 \tag{3.12}$$

and

$$X_0(U) = 1 \tag{3.13}$$

one finds

$$\left|\frac{I^{(G,F_C)}_{\lambda}(\beta)}{d(\lambda)}\right| < I^{(G,F_C)}_{0}(\beta) \qquad \text{if} \quad \lambda \neq 0 \tag{3.14}$$

So, in the limit $N \to \infty$,

$$Z = [I^{(G,F_C)}_0(\beta)]^N I^{(G,E)}_0(\beta) \tag{3.15}$$

$$\begin{aligned}\langle\{\sigma_{i+1/2}\}\rangle &= \frac{1}{Z} \left\{ \prod_i \left[\sum_{\lambda_{i+1/2}} \{\sigma_{i+1/2}, \lambda_{i+1/2}; 0\} I^{(G,F_C)}_{\lambda_{i+1/2}}(\beta) \right] \right\} I^{(G,E)}_0(\beta) \\ &= \frac{1}{Z} \left[\prod_i I^{(G,F_C)}_{\bar{\sigma}_{i+1/2}}(\beta) \right] I^{(G,E)}_0(\beta)\end{aligned}$$

or

$$\begin{aligned}\langle\{\sigma_{i+1/2}\}\rangle &= \prod_i \frac{I^{(G,F_C)}_{\bar{\sigma}_{i+1/2}}(\beta)}{I^{(G,F_C)}_0(\beta)} \\ &= \prod_i \langle X_{\sigma_{i+1/2}}(U_{i+1/2})\rangle \end{aligned} \tag{3.16}$$

Note that we have assumed that the observable is local, i.e., only a finite number of the $\sigma_{i+1/2}$ are nonzero.

Finally

$$\left\langle X_\sigma\left(\prod_i U_{i+1/2}\right)\right\rangle = \frac{1}{Z}\sum_\mu \{\sigma, \mu; 0\} I_\mu^{(G,E)}(\beta)[I_0^{(G,F_C)}]^N$$
$$= \frac{I_{\bar{\sigma}}^{(G,E)}(\beta)}{I_0^{(G,E)}(\beta)} \tag{3.17}$$

One can see that in this limit observables at different points are uncorrelated, and the theory is factorized into a matter part and a pure gauge part, just as is deduced from the previous theorems. Also note that if G is Abelian, the condition of conjugation invariance is automatically satisfied. So all of the results obtained in this section are valid.

4. EXAMPLES

4.1. Gauge-Invariant Ising Model

As the simplest case, consider

$$H_0 = -J\sum_i S_i S_{i+1} \tag{4.1}$$

where each S_i takes the values ± 1. Now, H_0 has global gauge symmetry under the action of gauge group Z_2. From this, one can construct the gauge-invariant Hamiltonian

$$H = -J\sum_i S_i U_{i+1/2} S_{i+1} - K\prod_i U_{i+1/2} \tag{4.2}$$

where the $U_{i+1/2}$ also take the values ± 1.

This group has only two representations: the defining representation, and the trivial one. So, from (3.6), we have

$$Z = \sum_{\lambda=0}^{1} [I_\lambda^{(Z_2)}(\beta J)]^N I_\lambda^{(Z_2)}(\beta K)$$

where

$$I_0^{(Z_2)}(x) := \frac{1}{2}\sum_{U=\pm 1} \exp(xU) = \cosh x \tag{4.3}$$

and

$$I_1^{(Z_2)}(x) := \frac{1}{2}\sum_{U=\pm 1} U\exp(xU) = \sinh x \tag{4.4}$$

So,

$$Z = \cosh^N(\beta J)\cosh(\beta K) + \mathrm{sing}^N(\beta J)\sinh(\beta K) \tag{4.5}$$

Using

$$\{\rho, \tau; \phi\} = \delta^{(2)}_{\phi,\rho+\tau} \tag{4.6}$$

where

$$\sigma^{(n)}_{m,l} := \begin{cases} 1 & m \equiv l \pmod{n} \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

we have

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{1}{Z}\left[I_0^{(Z_2)}(\beta K) \prod_i I^{(Z_2)}_{\sigma_{i+1/2}}(\beta J) + I_1^{(Z_2)}(\beta K) \prod_i I^{(Z_2)}_{1-\sigma_{i+1/2}}(\beta J)\right]$$

or

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{\cosh(\beta K)\cosh^N(\beta J)\tanh^s(\beta J) + \sinh(\beta K)\sinh^N(\beta J)\tanh^{-s}(\beta J)}{\cosh(\beta K)\cosh^N(\beta J) + \sinh(\beta K)\sinh^N(\beta J)} \tag{4.8}$$

where

$$s := \sum_i \sigma_{i+1/2} \tag{4.9}$$

Finally

$$\left\langle \prod_i U_{i+1/2} \right\rangle = \frac{\cosh(\beta K)\sinh^N(\beta J) + \sinh(\beta K)\cosh^N(\beta J)}{\cosh(\beta K)\cosh^N(\beta J) + \sinh(\beta K)\sinh^N(\beta J)} \tag{4.10}$$

For $N \to \infty$,

$$Z = \cosh(\beta K)\cosh^N(\beta J) \tag{4.11}$$

$$\langle\{\sigma_{i+1/2}\}\rangle = \tanh^s(\beta J) \tag{4.12}$$

$$\left\langle \prod_i U_{i+1/2} \right\rangle = \tanh(\beta K) \tag{4.13}$$

4.2. Gauge-Invariant Potts Model

$$H_0 = -2J \sum_i \left(\delta_{S_i, S_{i+1}} - \frac{1}{2}\right) \tag{4.14}$$

where each S_i is a nonnegative integer less than n. This Hamiltonian has global Z_n invariance. The gauge-invariant Hamiltonian is

$$H = -2J \sum_i \left[\delta^{(n)}_{S_{i+1}+U_{i+1/2}-S_i,0} - \frac{1}{2}\right] - E\left(\sum_i U_{i+1/2}\right) \tag{4.15}$$

where E is a periodic function on integers with period n. Now, Z_n has n representations, the characters of which are

$$X_\lambda(m) := \exp\left(\frac{i2\pi\lambda m}{n}\right) \tag{4.16}$$

We have

$$I_\lambda^{(Z_n,\delta)}(x) := \frac{1}{n}\sum_{m=0}^{n-1} \exp\left(-\frac{i2\pi\lambda m}{n}\right)\exp(x\delta_{m,0})$$

$$\Rightarrow \quad I_\lambda^{(Z_n,\delta)}(x) = \frac{\exp(x)-1}{n} + \delta_{\lambda,0} \tag{4.17}$$

So

$$Z = \left\{ I_0^{(Z_n,E)}(\beta)\left[1 + \frac{\exp(2\beta J)-1}{n}\right]^N + \left[\sum_{\lambda=1}^{n-1} I_{n-\lambda}^{(Z_n,E)}(\beta)\right]\left[\frac{\exp(2\beta J)-1}{n}\right]^N \right\} \exp(-N\beta J)$$

or

$$Z = \left(\frac{2}{n}\right)^N \left\{ I_0^{(Z_n,E)}(\beta)\left[\frac{n}{2}\exp(-\beta J) + \sinh(\beta J)\right]^N + \sinh^N(\beta J)\sum_{\lambda=1}^{n-1} I_{n-\lambda}^{(Z_n,E)}(\beta)\right\} \tag{4.18}$$

where we have

$$I_\lambda^{(Z_n,E)}(x) := \frac{1}{n}\sum_{m=0}^{n-1}\exp\left(-\frac{i2\pi\lambda m}{n}\right)\exp[xE(m)] \tag{4.19}$$

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{(2/n)^N}{Z}\sum_{\lambda=0}^{n-1}\left\{ I_{n-\lambda}^{(Z_n,E)}(\beta)\prod_i \times\left[\frac{n}{2}\exp(-\beta J)\delta_{\lambda,\sigma_{i+1/2}} + \sinh(\beta J)\right]\right\} \tag{4.20}$$

$$\left\langle X_\sigma\left(\prod_i U_{i+1/2}\right)\right\rangle = \frac{(2/n)^N}{Z}\left\{ I_{n-\sigma}^{(Z_n,E)}(\beta)\left[\frac{n}{2}\exp(-\beta J) + \sinh(\beta J)\right]^N + \sinh^N(\beta J)\sum_{\lambda=1}^{n-1} I_{n-\lambda-\sigma}^{(Z_n,E)}(\beta)\right\} \tag{4.21}$$

One can easily verify that these relations reduce to those of the Ising model if $n = 2$ and $E(x) = K\cos(x)$.

In the thermodynamic limit one obtains

$$Z = \left(\frac{2}{n}\right)^N I_0^{(Z_n,E)}(\beta)\left[\frac{n}{2}\exp(-\beta J) + \sinh(\beta J)\right]^N \tag{4.22}$$

$$\langle\{\sigma_{i+1/2}\}\rangle = \left[\frac{\sinh(\beta J)}{(n/2)\exp(-\beta J) + \sinh(\beta J)}\right]^s \tag{4.23}$$

$$\left\langle X_\sigma\left(\prod_i U_{i+1/2}\right)\right\rangle = \frac{I_{n-\sigma}^{(Z_n,E)}(\beta)}{I_0^{(Z_n,E)}(\beta)} \tag{4.24}$$

where we have

$$s := \sum_i (1 - \delta_{\sigma_{i+1/2},0}) \tag{4.25}$$

4.2. Gauge-Invariant Classical Planar Spin Model

Taking

$$H_0 = -J\sum_i \mathrm{Im}(S_i^* S_{i+1}) \tag{4.26}$$

where the S_i are phases, we see that the gauge group is $U(1)$. The gauge-invariant Hamiltonian is

$$H = -\sum_i \mathrm{Im}(S_i^* U_{i+1/2} S_{i+1}) - E\left(\prod_i U_{i+1/2}\right) \tag{4.27}$$

where E is a function from $U(1)$ to R. We parametrize the gauge group by ξ. There are (countably) infinite representations labeled by integers:

$$\hat{U}_\lambda(\xi) = \exp(i\lambda\xi) \tag{4.28}$$

We have

$$\begin{aligned} I_\lambda^{[U(1),J\,\mathrm{Im}]}(\beta) &:= \frac{1}{2\pi}\int_0^{2\pi} d\xi\, \exp(\beta J \sin\xi - i\lambda\xi) \\ &= i^{-\lambda} I_\lambda(\beta J) \end{aligned} \tag{4.29}$$

where $I_\lambda(x)$ is the modified Bessel function. We also have

$$\{\rho, \tau; \phi\} = \delta_{\phi,\rho+\tau} \tag{4.30}$$

So

$$Z = \sum_{\lambda=-\infty}^{+\infty} [i^{-\lambda} I_\lambda(\beta J)]^N I_{-\lambda}^{[U(1),E]}(\beta) \tag{4.31}$$

$$\langle\{\sigma_{i+1/2}\}\rangle=\frac{1}{Z}\sum_{\lambda=-\infty}^{+\infty}\left\{I_{-\lambda}^{[U(1),E]}(\beta)\prod_i\left[i^{-\lambda+\sigma_{1/2}}I_{\lambda-\sigma_{i+1/2}}(\beta J)\right]\right\} \tag{4.32}$$

$$\left\langle\exp\left[i\sigma\left(\sum_i\xi_{i+1/2}\right)\right]\right\rangle=\frac{1}{Z}\sum_{\lambda=-\infty}^{+\infty}\left[i^{-\lambda}I_\lambda(\beta J)\right]^N I_{-\lambda-\sigma}^{[U(1),E]}(\beta) \tag{4.33}$$

In the thermodynamic limit

$$Z=[I_0(\beta J)]^N I_0^{[U(1),E]}(\beta) \tag{4.34}$$

$$\langle\{\sigma_{i+1/2}\}\rangle=\prod_i\left[\frac{i^{\sigma_i+1/2}I_{\sigma_i+1/2}(\beta J)}{I_0(\beta J)}\right] \tag{4.35}$$

$$\left\langle\exp\left[i\sigma\left(\sum_i\xi_{i+1/2}\right)\right]\right\rangle=\frac{I_{-\sigma}^{[U(1),E]}(\beta)}{I_0^{[U(1),E]}(\beta)} \tag{4.36}$$

4.4. Z_2 Gauge Theory on Multiple-Orbit Matter Field Space

As our last example, consider a generalized form of gauge-invariant Ising model, where the S_i can take a set of values $\{a_m\}$ which are symmetric with respect to zero and have absolute values less than or equal to unity. The gauge-invariant Hamiltonian is of the form (4.2). Using the results of Section 1, we have

$$\begin{aligned}\mu(|S_i|,|S_{i+1}|)&:=\frac{1}{2}\{\exp[\beta J(S_iS_{i+1})]+\exp[-\beta J(S_iS_{i+1})]\}\\&=\cosh(\beta JS_iS_{i+1})\end{aligned} \tag{4.37}$$

$$Z_{\mathrm{m}}=\sum_{\{S_i\}}\cosh(\beta JS_iS_{i+1}) \tag{4.38}$$

So Z_{m} is (in the thermodynamic limit) the Nth power of the largest eigenvalue of the matrix M with the entries

$$M_{pq}:=\cosh(\beta Ja_pa_q) \tag{4.39}$$

For example,

$$\{a_m\}=\{0,\pm1\}$$
$$\Rightarrow\quad \lambda_{\max}=\frac{1+2\cosh(\beta J)+[4\cosh^2(J)-4\cosh(\beta J)+9]^{1/2}}{2} \tag{4.40}$$

$$\{a_m\}=\{\pm1/3,\pm1\}$$
$$\begin{aligned}\Rightarrow\quad \lambda_{\max}&=\cosh(\beta J)+\cosh(\beta J/9)\\&\quad+\{[\cosh(\beta J)-\cosh(\beta J/9)]^2+4\cosh^2(\beta J/3)\}^{1/2}\end{aligned} \tag{4.41}$$

In either case,

$$Z = \cosh(\beta K)(\lambda_{\max})^N \tag{4.42}$$

Finally, if the set of possible values of the S_i is the interval $[-1, 1]$, we must solve the eigenvalue problem

$$\lambda \Lambda(x) = \int_{-1}^{1} \cosh(\beta Jxy)\, \Lambda(y)\, dy \tag{4.43}$$

for its largest eigenvalue, and insert it in (4.42).

5. DOUBLE GAUGE FIELD THEORIES

There is a special case where the theory has more invariance properties. Consider the Hamiltonian

$$H_0 = -\sum_{\langle ij \rangle} F(S_i^{-1} S_j) \tag{5.1}$$

where the S_i themselves belong to a non-Abelian group G, and F is a real-valued class function on G. The Hamiltonian H_0 has two independent global symmetries:

$$S_i \to g_L S_i, \qquad S_i \to S_i g_R \tag{5.2}$$

where g_L and g_R are group elements. Introducing two gauge fields L and R, one can make both of these invariances local. The general form of gauge transformation is

$$\begin{aligned} S_i &\to (g_L)_i S_i (g_R)_i \\ L_{\langle ij \rangle} &\to (g_L)_i L_{\langle ij \rangle} (g_L)_j^{-1} \\ R_{\langle ij \rangle} &\to (g_R)_i^{-1} R_{\langle ij \rangle} (g_R)_j \end{aligned} \tag{5.3}$$

It is easy to see that

$$H_{\mathrm{m}} = -\sum_{\langle ij \rangle} F(S_i^{-1} L_{\langle ij \rangle} S_j R_{\langle ji \rangle}) \tag{5.4}$$

is gauge-invariant. We can also add class functions of Wilson loops to the above expression. In one dimension, one comes then to the Hamiltonian

$$H = -\sum_i F(S_i^{-1} L_{i+1/2} S_{i+1} R_{i+1/2}) - E\left(\overrightarrow{\prod_i} L_{i+1/2}, \overleftarrow{\prod_i} R_{i+1/2}\right) \tag{5.5}$$

where

$$L_{i+1/2} := L_{\langle i,i+1 \rangle} \tag{5.6}$$

$$R_{i+1/2} := R_{\langle i+1,i \rangle} \tag{5.7}$$

$$\overrightarrow{\prod_i} X_i := X_1 X_2 \cdots \tag{5.8}$$

$$\overleftarrow{\prod_i} Y_i := \cdots Y_2 Y_1 \tag{5.9}$$

By gauge-fixing, one can eliminate the S_i and come to

$$H_{\text{gf}} = -\sum_i F(L_{i+1/2} R_{i+1/2}) - E\left(\overrightarrow{\prod_i} L_{i+1/2}, \overleftarrow{\prod_i} R_{i+1/2}\right) \tag{5.10}$$

from which (using the appendix) we find

$$Z = \sum_\mu \left[\frac{I_\mu^{(G,F)}(\beta)}{d(\mu)}\right]^N I_{\bar{\mu},\bar{\mu}}^{(G,E)}(\beta) \tag{5.11}$$

where

$$\exp[\beta E(L,R)] =: \sum_{\lambda_L,\lambda_R} I_{\lambda_L,\lambda_R}^{(G,E)}(\beta) X_{\lambda_L}(L) X_{\lambda_R}(R) \tag{5.12}$$

The observables of this theory are

$$\prod_i [X_{\sigma_{i+1/2}}(S_i^{-1} L_{i+1/2} S_{i+1} R_{i+1/2})]$$

and

$$X_{\sigma_L}\left(\overrightarrow{\prod_i} L_{i+1/2}\right) X_{\sigma_R}\left(\overleftarrow{\prod_i} R_{i+1/2}\right)$$

In a manner similar to the previous cases, one finds

$$\langle\{\sigma_{i+1/2}\}\rangle = \frac{1}{Z} \sum_\mu \left\{ I_{\bar{\mu},\bar{\mu}}^{(G,E)}(\beta) \prod_i \left[\frac{1}{d(\mu)} \sum_{\lambda_{i+1/2}} \{\sigma_{i+1/2}, \lambda_{i+1/2}; \mu\} I_{\lambda_{i+1/2}}^{G,F)}(\beta)\right]\right\} \tag{5.13}$$

$$\left\langle X_{\sigma_L}\left(\overrightarrow{\prod_i} L_{i+1/2}\right) X_{\sigma_R}\left(\overleftarrow{\prod_i} R_{i+1/2}\right)\right\rangle$$
$$= \frac{1}{Z} \sum_{\lambda,\mu_L,\mu_R} \{\sigma_L, \mu_L; \bar{\lambda}\}\{\sigma_R, \mu_R; \bar{\lambda}\} I_{\mu_L,\mu_R}^{(G,E)}(\beta) \left[\frac{I_\lambda^{(G,F)}(\beta)}{d(\lambda)}\right]^N \tag{5.14}$$

We see that the matter field correlations are distance-independent. In the thermodynamic limit,

$$Z = [I_0^{(G,F)}(\beta)]^N I_{0,0}^{(G,E)}(\beta) \tag{5.15}$$

$$\langle\{\sigma_{i+1/2}\}\rangle = \prod_i \left[\frac{I_{\bar{\sigma}_{i+1/2}}(G,F)(\beta)}{I_0(G,F)(\beta)}\right] \tag{5.16}$$

$$\left\langle X_{\sigma_L}\left(\prod_i^{\rightarrow} L_{i+1/2}\right) X_{\sigma_R}\left(\prod_i^{\leftarrow} R_{i+1/2}\right)\right\rangle = \frac{I^{(G,E)}_{\bar{\sigma}_L,\bar{\sigma}_R}(\beta)}{I^{(G,F)}_{0,0}(\beta)} \tag{5.17}$$

In fact, in the thermodynamic limit we are faced with two decoupled interactions: a one-particle interaction and a two-particle one.

6. CONCLUSION

We have shown that a one-dimensional gauge theory for compact gauge groups is exactly solvable in the thermodynamic limit. In fact, in this limit the gauge degrees of freedom completely decouple from the matter degrees of freedom. The gauge part then reduces to a one-particle theory. If the gauge group acts transitively on matter field spaces, then the matter part also reduces to a noninteracting system. In such a case, all of the observables will be uncorrelated. We saw that there are certain cases where the correlators become distance-independent even for finite lattices. So far, there has been no phase transition in these theories, However, in a future paper, we will introduce a generalization of these theories which does exhibit a first-order phase transition.

APPENDIX

Here we introduce a couple of group identities used in the text. Take a compact group G, and let $X_\lambda(g)$ be the character (trace) of the element g in the (irreducible unitary) representation λ. We have

$$\begin{aligned} X_{\bar{\lambda}}(g) &= [X_\lambda(g)]^* \\ &= X_\lambda(g^{-1}) \end{aligned} \tag{A1}$$

where $\bar{\lambda}$ is the complex conjugate representation of λ. Moreover,

$$\int dg\, X_\lambda(hg) X_\mu(h'g^{-1}) + \frac{X_\lambda(hh')}{d(\lambda)}\sigma_{\lambda,\mu} \tag{A2}$$

where $d(\lambda)$ is the dimension of the representation λ.

Any complex-valued class function on G can be expressed as a linear combination of the characters:

$$F(g) = F(hgh^{-1}) \quad \Rightarrow \quad F(g) = \sum_\lambda A_\lambda^{(G,F)} X_\lambda(g) \tag{A3}$$

and

$$A_\lambda^{(G,F)} = \int dg\, F(g) X_\lambda(g^{-1}) \tag{A4}$$

In particular,

$$\exp[xF(g)] = \sum_{\lambda} I_{\lambda}^{(G,F)}(x) X_{\lambda}(g) \tag{A5}$$

$$I_{\lambda}^{(G,F)}(x) = \int dg \, \exp[xF(g)] \, X_{\lambda}(g^{-1}) \tag{A6}$$

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